NON-BACKTRACKING SPECTRUM OF RANDOM GRAPHS

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NORM OF SPARSE RANDOM MATRICES
Erdős-Rényi Random Graph

Let $A \in M_n(\mathbb{R})$ be a symmetric matrix with iid Bernoulli entries $(A_{xy})_{x \geq y}$ above the diagonal,

$$\mathbb{P}(A_{xy} = 1) = 1 - \mathbb{P}(A_{xy} = 0) = \frac{d}{n}.$$ 

Then $d$ is the average degree of a vertex

$$d = \mathbb{E} \sum_{y=1}^{n} A_{xy} = \mathbb{E} \deg(x) = \mathbb{E} \|A_x\|_2^2.$$ 

Eigenvalues of $A$

$$\lambda_1 \geq \cdots \geq \lambda_n.$$
Histogram of Eigenvalues

Single realization for $n = 1000$, $d = 5$ and $d = 20$. 
If $d = d_n \to \infty$, the empirical measure of the renormalized eigenvalues converges to the Wigner’s semi-circle law: for any real $t$, in probability,

$$\frac{1}{n} \sum_{k=1}^{n} 1(\lambda_k \geq t\sqrt{d}) \to \int_{t}^{\infty} f_{sc}(\lambda) d\lambda,$$

where

$$f_{sc}(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2} \mathbf{1}(|\lambda| \leq 2).$$

In particular, for any fixed $k$, with high probability,

$$\lambda_k \geq 2\sqrt{d}(1 - \varepsilon_n).$$
Previously known bounds

Krivelevich & Sudakov (2003): with high probability,

$$\lambda_1 \sim \max \left( d, \max_x \sqrt{\deg(x)} \right).$$

($a_n \sim b_n$ means $a_n = b_n(1 + \varepsilon_n)$).

Füredi & Komlós (1981), Vu (2005): if $d \gg (\log n)^4$, with high probability,

$$\max_{k \geq 2} |\lambda_k| \sim 2\sqrt{d}.$$

Feige & Ofek (2005), Chung & Radcliffe (2011): if $d \geq \varepsilon \log n$, with high probability,

$$\max_{k \geq 2} |\lambda_k| \leq C\varepsilon\sqrt{d}.$$
If the support of $A_y$ and $A_x$ are disjoint,

$$\lambda_2 \geq \min(\|A_x\|_2, \|A_y\|_2) = \min\left(\sqrt{\text{deg}(x)}, \sqrt{\text{deg}(y)}\right).$$

If $d \ll \log n$ then $\max_x \text{deg}(x) \gg d$ and this lower bound is sharp.

**Theorem (with Benaych-Georges & Knowles)**

If $d \ll \log n$, with high probability, for any $1 \leq k \leq n^{1-\epsilon}$,

$$\lambda_k \sim -\lambda_{n+1-k} \sim \max_x [k] \sqrt{\text{deg}(x)}.$$

(Extends to non-homogeneous $\text{Ber}(p_{xy})$ variables).
A lower bound for the spectral gap

**Theorem (With Benaych-Georges & Knowles)**

With high probability,

\[
\max_{k \geq 2} |\lambda_k| \leq \max_x \sqrt{\text{deg}(x)} \left( 2 + \frac{C}{\sqrt{d \wedge n^{\frac{1}{13}}}} \right).
\]

Note that if \( d \gg \log n \) then

\[
d \sim \max_x \text{deg}(x) \sim \min_x \text{deg}(x),
\]

(and \( \lambda_1 \sim d \)). As a corollary

\[
\max_{k \geq 2} |\lambda_k| \sim 2\sqrt{d}.
\]
Let $H \in M_n(\mathbb{C})$ be an hermitian random matrix with independent centered entries $(H_{xy})_{x \geq y}$ above the diagonal,

$$\max_x \mathbb{E} \sum_y |H_{xy}|^2 = 1 \quad \text{max } x, y \mathbb{E}|H_{xy}|^2 = \frac{\kappa}{n} \quad \text{a.s.} - \text{max } x, y |H_{xy}| \leq \frac{1}{q}.$$

The scalar $q$ controls the sparsity and $\kappa$ the inhomogeneity of $H$.

**Theorem (with Benaych-Georges & Knowles)**

If $C \leq q \leq n^{\frac{1}{13}} \kappa^{-\frac{1}{12}}$, with high probability,

$$\|H\| \leq \|H\|_{2 \rightarrow \infty} \left(2 + \frac{C}{q}\right),$$

with $\|H\|_{2 \rightarrow \infty} = \max_x \|H_x\|_2$. 

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**Norm of sparse random matrices**
Assume $A$ has entries independent $\text{Ber}(p_{xy})$ variables above the diagonal: inhomogeneous Erdős-Rényi graph. Set

$$d = \max_x \mathbb{E} \deg(x) = \max_x \sum_y p_{xy}.$$ 

As a corollary, (if $\max_{x,y} p_{xy}$ not much larger than $d/n$),

$$\|A - \mathbb{E}A\| \leq \max_x \sqrt{\deg(x)} \left( 2 + \frac{C}{\sqrt{d} \wedge n^{\frac{1}{13}}} \right).$$

This can be used to infer $\mathbb{E}A = (p_{xy})$ from a single observation of $A$ (Davis-Kahan Theorem: we can estimate the eigenspaces of $\mathbb{E}A$ whose eigenvalues are larger than $\|A - \mathbb{E}A\|$).
Large trace + First moment method of Fűredi & Komlós (1981)
Expected trace of high power

For simplicity, let
\[ H = \frac{A - \mathbb{E}A}{\sqrt{d}} \]
be the normalized adjacency of the Erdős-Rényi graph with average degree \( d \).

For even \( k \), we aim at a bound of the form
\[ \|H\|^k \leq \text{Tr}(H^k) \leq n(2 + \varepsilon_n)^k. \]

If \( k \gg \log n \), we would find
\[ \|H\| \leq n^{\frac{1}{k}}(2 + \varepsilon_n) = 2 + \varepsilon'_n. \]

From Markov’s inequality, it is sufficient to prove
\[ \mathbb{E}\text{Tr}(H^k) \leq n(2 + \varepsilon_n)^k. \]
**Expected trace of high power**

Recall $H = (A - EA)/\sqrt{d}$. With $\gamma_{k+1} = \gamma_1$,

$$\mathbb{E} \text{Tr}(H^k) = \sum_{\gamma=(\gamma_1, \ldots, \gamma_k)} \mathbb{E} \prod_{t=1}^k H_{\gamma_t \gamma_{t+1}}.$$

The summand is zero if the walk $\gamma$ visits $\{\gamma_t, \gamma_{t+1}\}$ only once.

If $\mathcal{W}_k$ is the set of unlabeled closed walks of length $k$ which visits each edge at least twice,

$$\mathbb{E} \text{Tr}(H^k) \leq \sum_{\gamma \in \mathcal{W}_k} \left(\frac{1}{\sqrt{d}}\right)^k \left(\frac{d}{n}\right)^{e(\gamma)} n^{v(\gamma)}.$$

**Diagram:**

- Circle 1: 3, 16
- Circle 2: 4, 7, 15
- Circle 3: 13, 14
- Circle 4
- Circle 5: 0, 10, 11
- Circle 6

$\gamma = (1, 1, 1, 2, 3, 4, 3, 2, 5, 4, 5, 4, 3, 6, 3, 2, 1)$, $e(\gamma) = 7$, $v(\gamma) = 6$. 
Path counting

\[ \mathbb{E} \text{Tr}(H^k) \leq n \sum_{\gamma \in \mathcal{W}_k} \left( \frac{1}{\sqrt{d}} \right)^{k-2e(\gamma)} \left( \frac{1}{n} \right)^{e(\gamma)-v(\gamma)+1}. \]

Needed: bounds on the number of closed walks visiting each edge at least twice counted by the genus \( g = e - v + 1 \) and the number of edges of the graph they span.

If \( e = k/2 \) and \( v = k/2 + 1 \), each edge is visited twice and the graph is a tree. The number of such walks is the Catalan number \( C_{k/2} = (2 + \varepsilon_n)^k \).

Sharpest bound obtained by Vu (2005), works for \( d \gg (\log n)^4 \).
Non-backtracking matrix
Let $H$ be a matrix in $M_n(\mathbb{C})$. Consider the matrix $B$ in $M_{n^2}(\mathbb{C})$ with entries

$$B_{ef} = H_{ab} \mathbf{1}(y = a)\mathbf{1}(x \neq b).$$

where $e = (x, y)$ and $f = (a, b)$.

Beware that if $H$ is hermitian, $B$ is not! (not even normal).

Hashimoto (1989).
Spectral bound

Recall

\[ \|H\|_{2 \to \infty} = \max_x \sqrt{\sum_y |H_{xy}|^2} \quad \text{and} \quad \|H\|_{1 \to \infty} = \max_{x,y} |H_{xy}|. \]

**Theorem (with Benaych-Georges & Knowles)**

If \( H \) is hermitian with non-backtracking matrix \( B \), then

\[ \|H\| \leq 2\|H\|_{2 \to \infty} + \left( \rho(B) - \|H\|_{2 \to \infty} \right)^2 \frac{2}{\|H\|_{2 \to \infty}} + C\|H\|_{1 \to \infty}. \]
An extended Ihara-Bass formula

Lemma (Watanabe & Fukumizu (2009))

Let $H$ be hermitian with nonbacktracking matrix $B$ and let $\lambda \in \mathbb{C}$, $\lambda \neq |H_{xy}|$ for all $x, y$. Define $H_\lambda$ and $D_\lambda$ diagonal

$$(H_\lambda)_{xy} = \frac{H_{xy}}{1 - \lambda^{-2}|H_{xy}|^2} \quad (D_\lambda)_{xx} = \lambda + \frac{1}{\lambda} \sum_y \frac{|H_{xy}|^2}{1 - \lambda^{-2}|H_{xy}|^2}.$$ 

Then $\lambda \in \sigma(B)$ if and only if $0 \in \sigma(H_\lambda - D_\lambda)$.

Key Fact: Assume $\|H\|_{2\rightarrow\infty} = 1$.

If $\lambda + 1/\lambda$ is in $\sigma(H)$ with $\lambda \geq 1 + C \sqrt{\|H\|_{1\rightarrow\infty}}$ then there exists real $\lambda' \geq \lambda - C \sqrt{\|H\|_{1\rightarrow\infty}}$ in $\sigma(B)$. 
Spectral radius of random nonbacktracking matrices

Let $H \in M_n(\mathbb{C})$ be an hermitian random matrix with independent centered entries $(H_{xy})_{x \geq y}$ above the diagonal,

$$\max_x \mathbb{E} \sum_y |H_{xy}|^2 = 1 \quad \max_{x,y} \mathbb{E}|H_{xy}|^2 = \frac{\kappa}{n} \quad a.s. - \max_{x,y} |H_{xy}| \leq \frac{1}{q}.$$  

**Theorem (with Benaych-Georges & Knowles)**

If $C \leq q \leq n^{\frac{1}{13}} \kappa^{-\frac{1}{12}}$, with high probability,

$$\rho(B) \leq 1 + \frac{C}{q}.$$  

As a corollary, we obtain that with high probability,

$$\|H\| \leq \|H\|_{2 \rightarrow \infty} \left( 2 + \frac{C}{q} \right).$$
**Expected trace of high power**

For simplicity, let

$$H = \frac{A - \mathbb{E}A}{\sqrt{d}}$$

be the normalized adjacency of the Erdős-Rényi graph with average degree $d$.

For even $k$,

$$\rho(B)^k \leq \| B^{k/2} (B^{k/2})^* \| \leq \text{Tr} \left( B^{k/2} (B^{k/2})^* \right).$$

We aim at, for some $k \gg \log n$,

$$\mathbb{E} \text{Tr} \left( B^{k/2} (B^{k/2})^* \right) \leq n^2 (1 + \varepsilon)^k.$$
Expected trace of high power

Expanding the trace

\[ \mathbb{E} \text{Tr} \left( B^{k/2} (B^{k/2})^* \right) \leq n^2 \sum_{\gamma \in \mathcal{N}_k} \left( \frac{1}{\sqrt{d}} \right)^{k-2e(\gamma)} \left( \frac{1}{n} \right)^{e(\gamma)-v(\gamma)+1} , \]

where \( \mathcal{N}_k \) is the set of unlabeled walks \( \gamma = (\gamma_1, \ldots, \gamma_k) \) which visits each edge at least twice,

\[ \gamma_{t+1} \neq \gamma_{t-1} \quad \text{for all} \quad t \neq \frac{k}{2}, \]

and the boundary conditions

\[ \gamma_1 \]
\[ \gamma_{\frac{k}{2}} \]
\[ \gamma_{\frac{k}{2}+1} \]
\[ \gamma_{k-1} \]
\[ \gamma_k \]
Path counting

Let $\gamma$ in $\mathcal{N}_k$ which visits $e \leq k/2$ edges and $v$ vertices. Set $g = e - v + 1 \geq 0$.

The walk in the reduced graph determines the original walk.

The reduced graph has at most $3g + 1$ edges, $2g + 2$ vertices and has maximal degree $2g + 1$.

We can then count walks in $\mathcal{N}_k$ in terms of $g$ and $k - 2e$.

Extremal Eigenvalues of Diluted Random Matrices
**Regular graph**

Consider the adjacency matrix $A$ of a $d$-regular graph on $n$ vertices with eigenvalues

$$d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

**Alon-Boppana lower bound:**

$$\lambda_2 \geq 2\sqrt{d - 1} - \frac{C_d}{(\log n)^2}.$$

A (non-bipartite) graph is **Ramanujan** if $\lambda_2 \lor (-\lambda_n) \leq 2\sqrt{d - 1}$. 
Alon’s Conjecture (1986)

Theorem (Friedman (2008))

Fix integer \(d \geq 3\). Let \(G_n\) is a sequence of uniformly distributed \(d\)-regular graphs on \(n\) vertices, then with high probability,

\[
\lambda_2 = 2\sqrt{d-1} + \varepsilon_n = -\lambda_n.
\]

Most regular graphs are nearly Ramanujan!!
For even $k \gg \log n$, we aim at

$$\mathbb{E}\text{Tr}(A^k) - d^k = \mathbb{E}\text{Tr}\left( A - \frac{d}{n}11^* \right)^k \leq n\left( 2\sqrt{d-1} + \varepsilon \right)^k$$

This is wrong!

The probability that the graph contains $K_{d+1}$ as subgraph is at least $n^{-c}$. On this event $\lambda_2 = d$. Hence, for even $k \gg \log n$,

$$\mathbb{E}\text{Tr}\left( A - \frac{d}{n}11^* \right)^k \geq n^{-c}d^k \gg n\left( 2\sqrt{d-1} + \varepsilon \right)^k.$$ 

Subgraphs which have polynomially small probability compromise the first moment method. Called Tangles.
A new strategy (2015)

1. Use the non-backtracking matrix $B$ of $A$ and use Ihara-Bass formula \textit{(Friedman (2008))}.

2. Remove the tangles in $B^\ell$ with $\ell = c \log n$ well chosen: count only non-backracking paths which visit at most one cycle.

3. Project on $1\perp$: bound $|\mu_2|$ in terms of a matrix with centered entries + remainder \textit{(Massoulié (2014))}
\[
|\mu_2|^\ell \leq \sup_{g \perp 1} \frac{\|B^\ell g\|_2}{\|g\|_2} \leq \|B^\ell\| + \frac{\|R^\ell\|}{n}.
\]

4. Use the trace method + first moment method to evaluate the norm of these matrices. \textit{Beware that the entries of $A$ are not independent.}
Non-backtracking version of Alon’s conjecture

Let $B$ be the non-backtracking matrix of the adjacency matrix $A$, with eigenvalues

$$d - 1 = \mu_1 \geq |\mu_2| \geq \cdots$$

$$\sigma(B) = \{\pm 1\} \cup \{\mu : \mu^2 - \lambda \mu + (d - 1) = 0, \lambda \in \sigma(A)\}.$$ 

Theorem (Friedman (2008))

Fix integer $d \geq 3$. Let $G_n$ be a sequence of uniformly distributed $d$-regular graphs on $n$ vertices, then with high probability,

$$|\mu_2| \leq \sqrt{d - 1} + \varepsilon_n.$$
Non-backtracking spectrum of Erdős-Rényi graphs

Eigenvalues of $B$ for an Erdős-Rényi graph with average degree $d = 4$ and $n = 500$ vertices.
Erdős-Rényi Graph

Let $B$ be the non-backtracking matrix of the adjacency matrix $A$, with eigenvalues

$$
\mu_1 \geq |\mu_2| \geq \cdots
$$

Theorem (with Massoulié & Lelarge)

Let $d > 1$ and $G_n$ be an Erdős-Rényi graph with average degree $d$. With high probability,

$$
\mu_1 = d + \varepsilon_n
$$

$$
|\mu_2| \leq \sqrt{d} + \varepsilon_n.
$$

(Extends to stochastic block models with constant average degree).
Random Lifts
Consider the hermitian matrix in $M_{k \times n}(\mathbb{C})$,

$$A = a_0 \otimes I + \sum_{i=1}^{d} a_i \otimes S_i + a_i^* \otimes S_i^{-1},$$

where $S_i \in M_n(\mathbb{C})$ are independent uniform permutations matrices and $a_i \in M_k(\mathbb{C})$ with $a_0^* = a_0$.

If $k = 1$, $a_0 = 0$ and all $a_i = 1$, this is the adjacency matrix of a random $2d$-regular graph. If $k = 1$, $a_0 = 0$, $a_i \geq 0$, $2 \sum_i a_i = 1$ anisotropic random walk on a random graph.
If $a_0 = 0$ and $a_i = E_{x_i,y_i}$ then

$$A_1 = \sum_{i=1}^{d} (a_i + a_i^*).$$

is the adjacency matrix of a graph with $k$ vertices and $d$ edges.
Random Lifts

If $a_0 = 0$ and $a_i = E_{x_i, y_i}$ then

$$A_1 = \sum_{i=1}^{d} (a_i + a_i^*).$$

is the adjacency matrix of a graph with $k$ vertices and $d$ edges.
NEW EIGENVALUES

\[ A = a_0 \otimes I + \sum_{i=1}^{d} a_i \otimes S_i + a_i^* \otimes S_i^{-1}, \]

The vector space \( \mathbb{C}^k \otimes 1 \)

is invariant, hence all eigenvalues of \( A_1 \) are eigenvalues of \( A \).

The largest eigenvalue of \( A \) not in \( A_1 \) is

\[ r\left( A|_{(\mathbb{C}^k \otimes 1)^\perp} \right) = \sup_{f \in (\mathbb{C}^k \otimes 1)^\perp} \frac{\langle f, Af \rangle}{\| f \|^2_2}. \]
New eigenvalues

Simulation with $d = 2$, $n = 500$, $k = 1$ and for some anisotropic random walk.
**Limit Operator**

In \( C^k \otimes \ell^2(X) \), where \( X = \mathbb{Z} \ast \cdots \ast \mathbb{Z} \) is the free product of \( d \) copies of \( \mathbb{Z} \) (i.e. 2d-infinite regular tree), let

\[
A_* = a_0 \otimes 1 + \sum_{i=1}^{d} a_i \otimes \lambda(g_i) + a_i^* \otimes \lambda(g_i^{-1}),
\]

where \((g_1, \cdots, g_d)\) are the generators of the copies of \( \mathbb{Z} \) and \( \lambda(\cdot) \) is the left regular representation (left multiplication).
The right edge of the spectrum is

\[ r(A_\star) = \sup \{\lambda : \lambda \in \sigma(A_\star)\}. \]

It has an explicit expression. For \( k = 1 \), Akemann-Ostrand (1976)

\[ r(A_\star) = \inf_{s > 0} \left( a_0 + 2s + 2 \sum_{i=1}^{d} \left( \sqrt{s + |a_i|^2} - s \right) \right). \]

For \( k \geq 1 \), if \( a_i^* = a_i \), Lehner (1999).
Generalized Alon’s Conjecture

\[ A = a_0 \otimes I + \sum_{i=1}^{d} a_i \otimes S_i + a_i^* \otimes S_i^{-1} \]

\[ r(T) = \sup_{f} \frac{\langle f, Tf \rangle}{\|f\|_2^2}. \]

Theorem (with Collins)

If \( S_1, \cdots, S_d \) are iid uniform permutations in \( S_n \), in probability,

\[ r\left( A_{|(\mathbb{C}^k \otimes 1)\perp} \right) \rightarrow r(A_\star). \]

Strong asymptotic freeness : as a corollary, for any non-commutative polynomial \( P \), in probability,

\[ \| P(S_1, \cdots, S_d)_{1\perp} \| \rightarrow \| P(\lambda(g_1), \cdots, \lambda(g_d)) \|. \]
With $i^* = i + d$, $i^{**} = i$, $a_{i^*} = a_i^*$, $S_{i^*} = S_i^*$,

\[ A = a_0 + \sum_{i=1}^{2d} a_i \otimes S_i \]

\[ B = \sum_{i \neq j^*} a_j \otimes S_i \otimes E_{ij}. \]

Again, $B$ decomposes $\bot$ on $(\mathbb{C}^k \otimes 1 \otimes \mathbb{C}^{2d}) \oplus (\mathbb{C}^k \otimes 1 \otimes \mathbb{C}^{2d})^\bot$ and all eigenvalues of $B_1 = \sum_{i \neq j^*} a_j \otimes E_{ij}$ are eigenvalues of $B$.

**Theorem (with Collins)**

Let $\varepsilon > 0$ and $S_1, \ldots, S_d$ be iid uniform permutations in $S_n$.

With high probability, for all $a_1, \ldots, a_d \in M_k(\mathbb{C})$ with $\|a_i\| \leq 1,$

\[ \rho(B|_{(\mathbb{C}^r \otimes 1 \otimes \mathbb{C}^{2d})^\bot}) \leq \rho(B^\star) + \varepsilon, \]

with $B$ and $B^\star$ nonbacktracking operators of $A$ and $A^\star$. 
Non-backtracking version

Simulation with $d = 2$, $n = 500$, $k = 1$ and for some anisotropic random walk.
1. Convergence of spectral radii for all weighted non-backtracking matrices implies convergence of right edges of $A$.
Based on an extension of Ihara-Bass formula (analog to Anantharaman (2016)).

2. Follow and adapt the strategy for $k = 1$ and all $a_i = 1$.

(Works also if some permutations are uniform matchings ($n$ even)).
Concluding Words
Non-backtracking operators are a powerful tool to study the edge of the spectrum in locally tree-like graphs.

It can also be used inside the spectrum, Anantharaman & Sabri (2017).
Perspectives

- Extremal eigenvalues for other ensembles of random graphs, such as configuration model?

- Equivalent of $\lambda_2$ for Erdős-Rényi graph when $d$ of order $\log n$?

- Fluctuations of extremal eigenvalues?

- Alon-Boppana lower bound for non-backtracking matrices?
Thank you for your attention!