Point processes and $q$-hypergeometric polynomials

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Point processes

Definition: A (simple) **point configuration** on a locally compact space space $\mathcal{X}$ is a subset $X \subset \mathcal{X}$ without accumulation points. In particular, $X$ must be either finite or countable.

Definition: A (simple) **point process** on $\mathcal{X}$ is determined by a probability measure $M$ on the space of point configurations.

Thus, $M$ gives rise to an ensemble of **random** point configurations.
Examples of $N$-point processes

$\mathcal{X} := \mathbb{R}$, $N = 1, 2, 3, \ldots$, $W(x) \geq 0$ a function on $\mathbb{R}$, 

$X = (x_1, \ldots, x_N)$, $x_1 > \cdots > x_N$, 

\[
M_N(dX) = \text{const} \cdot \prod_{1 \leq i < j \leq N} (x_i - x_j)^{2\theta} \cdot \prod_{i=1}^{N} W(x_i)dx_i.
\]
Large-$N$ limit

Problem: Construct infinite-point processes out of $N$-point processes.

Want to get a probability measure $M_\infty$ living on infinite configurations via a limit transition

$$M_\infty = \lim_{N \to \infty} M_N$$

The case $\theta = 1$ is distinguished as it leads to exactly solvable models.

Example: $\text{GUE}_N (\theta = 1, W(x) = e^{-x^2/2}) \to$ sine process.

General $\theta > 0$: recent works based on the tridiagonal model of random matrices.
Another approach

• Origin: representation theory of big groups.
• Totally different method (applicable in non exactly solvable cases)
• Different class of $N$-point models
The classic Beta distribution

on $I := [0, 1]$ or $I := [-1, 1]$

$$W(x)dx = \frac{\Gamma(A + 1)\Gamma(B + 1)}{\Gamma(A + B + 2)} x^A (1 - x)^B dx, \quad 0 \leq x \leq 1$$

or

$$W(x)dx = \frac{\Gamma(A + 1)\Gamma(B + 1)}{2^{A+B+1} \Gamma(A + B + 2)} (1-x)^A (1+x)^B dx, \quad -1 \leq x \leq 1$$

$A > -1, \quad B > -1.$
The $N$-dimensional Beta distribution

$X = (x_1 > \cdots > x_N)$ ranges over $[0, 1]$ or $[-1, 1]$,

$$M_N(dX) = \frac{1}{Z_N} \cdot \prod_{1 \leq i < j \leq N} |x_i - x_j|^{2\theta} \cdot \prod_{i=1}^{N} W(x_i) dx_i$$

The value of $Z_N$ is Selberg’s integral.
Quantization (that is, $q$-version) of interval $[-1, 1]$

Fix $q \in (0, 1)$. The $q$-version of $[-1, 1]$ is

$$[-1, 1]_q := \{-1, -q, -q^2, \ldots\} \cup \{\ldots, q^2, q, 1\}$$

Notation from $q$-calculus

$$(y; q)_\infty := (1 - y)(1 - yq)(1 - yq^2) \ldots.$$
Andrews-Askey’s $q$-version of Beta distribution

It is a discrete probability distribution on $[-1, 1]_q$ with weights

$$W(x) = \frac{1}{\mathcal{Z}} |x| \frac{(xq, q)_\infty (-xq; q)_\infty}{(cx; q)_\infty (dx; q)_\infty}, \quad x \in [-1, 1]_q.$$ 

Conditions on parameters:

• either (principal series) $d = \bar{c}, \ c \in \mathbb{C} \setminus \mathbb{R}$
• or (complementary series) $c, d \in \mathbb{R}$ are such that $[c, d]$ lies inside an interval between two neighboring points from

$$\ldots, \ -q^{-2}, \ -q^{-1}, \ -1, \ 1, \ q^{-1}, \ q^{-2}, \ \ldots$$

Degeneration: if $c = q^{A+1}, \ d = -q^{B+1}$ and $q \to 1^-$, then

$q$-Beta $\to$ classic Beta
$N$-dimensional $(q, t)$-Beta distribution

Introduce second (Macdonald) parameter $t = q^\theta$, $\theta > 0$. For simplification assume $\theta \in \{1, 2, 3, \ldots \}$

$X = (x_1 > \cdots > x_N) \subset [-1, 1]_q$.

\[
M_N(X; c, d) = \frac{1}{Z_N} \left| \prod_{1 \leq i \neq j \leq N} \prod_{r=0}^{\theta-1} (x_i - x_j q^r) \right| \times \prod_{i=1}^{N} x_i \left( \frac{(x_i q; q)_\infty (-x_i q; q)_\infty}{(cx; q)_\infty (dx; q)_\infty} \right)
\]
Comments

Let $\theta = 1$. Then:
• the product over $(i, j)$ reduces to $\prod_{i<j}(x_i - x_j)^2$;
• all $N$-point configurations $X$ have nonzero (hence strictly positive) weights.

Let $\theta = 2, 3, \ldots$. Then:
• the product over $(i, j)$ is a ‘quantized’ version of $\prod_{i<j}(x_i - x_j)^{2\theta}$;
• the weight of $X$ is nonzero if and only if $X$ is ‘$\theta$-sparse’ meaning that any two points of $X$ are separated by at least $\theta - 1$ unoccupied nodes of $[-1, 1]_q$. 

\[ x_{i+1} \quad \geq \theta-1 \text{ nodes} \quad x_i \]
Infinite-dimensional \((q, t)\)-Beta distribution

Fix parameters \((\gamma, \delta)\) which are either in the principal series or are in the complementary series and have the same sign. We set \((c, d) = (\gamma t^{1-N}, \delta t^{1-N})\), so that they vary together with \(N\).

**Main Theorem.** There exists a limit

\[
M_{\gamma,\delta} = \lim_{N \to \infty} M_N(\cdot; \gamma t^{1-N}, \delta t^{1-N}).
\]

It is a probability measure living on the space

\[
\Omega_{\infty} := \{\text{infinite-point } \theta\text{-sparse configurations on } [-1, 1]_q\}.
\]

**Comments:**

1. No scaling of space.
2. Infinite configurations are locally finite because 0 is excluded.
3. As \(q \to 1^-\), the construction is destroyed. Hence, it is purely ‘quantum’ phenomenon.
The method of intertwiners

Notation: \( q \in (0, 1), t = q^\theta, \theta \in \mathbb{Z}_{\geq 1} \)

\( \Omega_N := \{ N\text{-point } \theta\text{-sparse configurations on } [-1, 1]_q \} \)

We construct certain stochastic matrices \( \Lambda_{N-1}^N = [\Lambda_{N-1}^N(X, Y)] \) of format \( \Omega_N \times \Omega_{N-1} \), depending only on \((q, t)\).

**Theorem 1.** Abbreviate \( M_N = M_N(\cdot, ; \gamma t^{1-N}, \delta t^{1-N}) \). We have

\[
M_N \Lambda_{N-1}^N = M_{N-1}, \quad N = 2, 3, \ldots .
\]

In more detail, let \( Y = (y_1 > \cdots > y_{N-1}) \in \Omega_{N-1} \) be fixed and \( X = (x_1 > \cdots > x_N) \) range over \( \Omega_N \). Then

\[
\sum_{X \in \Omega_N} M_N(X) \Lambda_{N-1}^N(X, Y) = M_{N-1}(Y).
\]
Theorem 2. (i) There exists a bijective correspondence

\[ \{ \mathcal{M}_N : N = 1, 2, \ldots \} \leftrightarrow \mathcal{M}_\infty \]

between sequences of probability measures \( \mathcal{M}_N \in \text{Prob}(\Omega_N) \) subject to relations

\[ \mathcal{M}_N \Lambda^N_{N-1} = \mathcal{M}_{N-1}, \quad N = 2, 3, \ldots, \]

and probability measures \( \mathcal{M}_\infty \in \text{Prob}(\Omega_\infty) \)

(ii) Under this correspondence, \( \mathcal{M}_\infty = \lim_{N \to \infty} \mathcal{M}_N \).

The space \( \Omega_\infty \) arises as the entrance boundary of the Markov chain

\[ \Omega_1 \leftarrow \Omega_2 \leftarrow \Omega_3 \leftarrow \ldots \]

We call \( \mathcal{M}_\infty \) the boundary measure of \( \{ \mathcal{M}_N \} \).

\bullet This abstract result is similar to Fatou's theorem for boundary values of harmonic functions.
Reformulation of (i):

Consider the projective limit space \( \lim_{\leftarrow} \text{Prob}(\Omega_N) \), where the maps \( \text{Prob}(\Omega_{N-1}) \leftarrow \text{Prob}(\Omega_N) \) are determined by the matrices \( \Lambda_{N-1}^N \).

There exist Markov kernels \( \Lambda_N^\infty \) linking \( \Omega_\infty \) to the spaces \( \Omega_N \), which establish a bijection

\[
\lim_{\leftarrow} \text{Prob}(\Omega_N) \leftarrow \text{Prob}(\Omega_\infty).
\]
Proof of Main Theorem:

By virtue of Theorem 1, the sequence $\mathcal{M}_N := M_N(\cdot; \gamma^t \delta^t)$ satisfies the assumption of Theorem 2. Applying it, we obtain the corresponding boundary measure $\mathcal{M}_\infty$ on $\Omega_\infty$. It is the desired limit measure $M_\infty$.

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Summary

We consider a sequence of 1-, 2-, ..., $N$-, ... point processes on $[-1, 1]_q$. They depend on parameters $q, t = \theta, \gamma, \delta$ and are determined by atomic probability measures $M_N(X; \gamma^t \delta^t)$, which are lattice analogues of the multidimensional Beta distributions. The main result is a rigorous construction of the large-$N$ limit process. It is governed by a probability measure $M_\infty^\gamma, \delta$ on a space $\Omega_\infty$ of infinite configurations. No scaling is required. The key point is the existence of Markov kernels (stochastic matrices) $\Lambda_{N-1}^N$ that intertwine the pre-limit processes. This is in contrast with approaches used in RMT.
The special case $\theta = 1$

For $\theta = 1$ the point process with law $M_{\infty, \delta}^\gamma$ is determinantal. This means that there exists a function $K(x, y)$ on $[-1, 1]_q \times [-1, 1]_q$ such that for any $n = 1, 2, \ldots$ and any $n$-tuple $x_1, \ldots, x_n \in [-1, 1]_q$,

$$\Pr\{\text{random } X \text{ contains } x_1, \ldots, x_n\} = \det[K(x_i, x_j)]_{i,j=1}^n.$$

Next, for the kernel $K(x, y)$, there is an explicit expression through the $q$-hypergeometric series

$$2\phi_1 \left[ \begin{array}{c} A, B \\ C \end{array} \right| q, z \right] := 1 + \sum_{n=1}^{\infty} \frac{(A; q)_n (B; q)_n}{(C; q)_n (q; q)_n} z^n,$$

$$(A; q)_n := (1 - A)(1 - Aq) \ldots (1 - Aq^{n-1}).$$

In this sense, the case $\theta = 1$ is exactly solvable. (The result follows from [Gorin & O., 2016].)
What about general $\theta$?

When $\theta \neq 1$, there is no hope that the measure $M^\gamma,\delta_\infty$ is determinantal. What then can be computed in an explicit form?

- The classic Beta distribution on $[-1, 1]$ is the weight measure for the classic Jacobi orthogonal polynomials;

- Likewise, the $q$-Beta distribution on $[-1, 1]_q$ serves as the weight measure for a system of orthogonal polynomials — the **big $q$-Jacobi polynomials** (Andrews & Askey, 1984), which are expressed through the $q$-hypergeometric series $\,\,_{3}\phi_{2}$.

- It turns out that one can define an **infinite-dimensional version** of big $q$-Jacobi polynomials and explicitly compute them.

- This provides a **characterization** of measures $M^\gamma,\delta_\infty$ with general parameter $\theta$. 
**Algebra Sym**

Definition: The algebra of symmetric functions is the algebra of formal power series in countably many variables, invariant under permutations, and of bounded degree

$$\text{Sym} \subset \mathbb{R}[[x_1, x_2, \ldots]].$$

It is freely generated by the power sums

$$x_1^n + x_2^n + x_3^n + \ldots, \quad n = 1, 2, \ldots.$$

**Easy Lemma.** Elements $F \in \text{Sym}$ can be evaluated at configurations $X \in \Omega_\infty$. (Proof: $\sum_{x \in [-1,1]_q} |x|^n < \infty$ for each $n = 1, 2, \ldots$.)

**Corollary.** Sym is embedded into $C(\Omega)$, with dense image, where

$$\Omega := \{\text{all configurations on } [-1, 1]_q \} = \Omega_\infty \cup \bigcup_{N=0}^{\infty} \Omega_N.$$
Characterization of measures $M_{\gamma,\delta}$

Let $\lambda$ range over the set $\mathbb{Y}$ of partitions (= Young diagrams).

$\{P_\lambda(\cdot; q, t)\}$: the Macdonald symmetric functions

One can explicitly construct a basis $\{\Phi_{\gamma,\delta}^{\lambda} : \lambda \in \mathbb{Y}\} \subset \text{Sym}$ s.t.:

- $\Phi_{\lambda}^{\gamma,\delta} = P_\lambda(\cdot; q, t) + \text{lower degree terms}$. In particular, $\Phi_{0}^{\gamma,\delta} = 1$.
- The corresponding functions $\Phi_{\lambda}^{\gamma,\delta}(X)$ on $\Omega$ form an orthogonal basis in the Hilbert space $L^2(\Omega, M_{\gamma,\delta}^{\infty})$.

**Corollary.** Once $\{\Phi_{\lambda}^{\gamma,\delta}\}$ is exhibited, $M_{\gamma,\delta}^{\infty}$ may be characterized by

$$\int_{\Omega} \Phi_{\lambda}^{\gamma,\delta}(X) M_{\gamma,\delta}^{\infty}(dX) = 0 \quad \text{for all nonzero } \lambda \in \mathbb{Y}.$$ 

- The very existence of $\{\Phi_{\lambda}^{\gamma,\delta}\}$ is not at all evident. It means that the system $\{P_\lambda(\cdot; q, t) : \lambda \in \mathbb{Y}\}$ of Macdonald symmetric functions admits orthogonalization with respect to $M_{\gamma,\delta}^{\infty}$. 
Motivation from representation theory

$G$: compact group; $\hat{G}$: the set of its irreducible representations; $\hat{G}$ is a countable set; it is called the **dual object** to $G$.

**Question:** For $H \to G$, how to define a dual "morphism" $\hat{G} \to \hat{H}$?

For noncommutative groups there is no natural dual map

**Solution:** $\hat{G} \to \hat{H}$ is a Markov kernel $\Lambda^G_H$ (stochastic matrix) of format $\hat{G} \times \hat{H}$.

**Definition by example:** $G := U(N)$, $H := U(N-1)$

$$\hat{G} \leftrightarrow \{ \lambda : \lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N \},$$

$$\hat{H} \leftrightarrow \{ \mu : \mu = (\mu_1 \geq \cdots \geq \mu_{N-1}) \in \mathbb{Z}^{N-1} \}$$

$T^N_\lambda, T^{N-1}_\mu$: the corresponding irreps of $G$ and $H$, respectively.
Prototype of stochastic matrices $\Lambda_{N-1}^N$

Gelfand–Tsetlin branching rule:

$$ T_N^\lambda \downarrow U(N-1) = \bigoplus_{\mu: \mu \prec \lambda} T_{\mu}^{N-1}, $$

where $\mu \prec \lambda$ means $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq \mu_{N-1} \geq \lambda_N$.

Counting dimensions,

$$ \dim T_N^\lambda = \sum_{\mu: \mu \prec \lambda} \dim T_{\mu}^{N-1}. $$

This suggests

$$ \Lambda_{U(N-1)}^{U(N)}(\lambda, \mu) := \begin{cases} \dim T_{\mu}^{N-1} / \dim T_N^\lambda, & \mu \prec \lambda, \\ 0, & \text{otherwise} \end{cases} $$

This is a prototype of the matrices $\Lambda_{N-1}^N$ linking the configuration spaces $\Omega_N$ and $\Omega_{N-1}$. 
References

